Geographic Information System

Spatial Interpolation

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Photo credit: ChatGPT

Outline

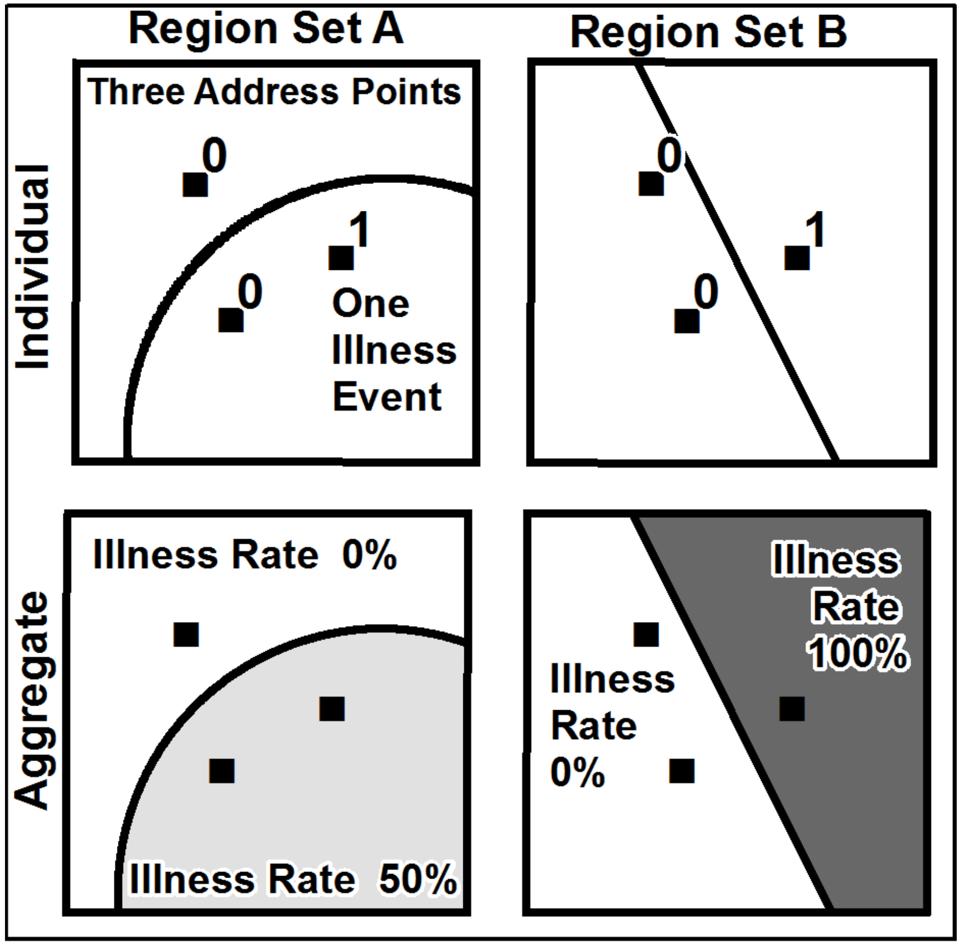
- Gridding
- MAUP
- Spatial Interpolation
- Inverse Distance Weighting
- Kriging
- Natural Neighbor
- Spline

Gridding

- Gridding (fishnet) is a useful method for transforming from point to polygon data type; meanwhile, we usually use gridding to quantify the population data to a spatial extent.
- Secondly, we usually use the gridding method to divide a map into equal-area rectangles or squares, which can help with location referencing and communication.
- Do you know any spatial data using gridding division?

MAUP

- The modifiable areal unit problem (MAUP) is a source of statistical bias that can significantly impact the results of statistical hypothesis tests. MAUP affects results when point-based measures of spatial phenomena are aggregated into spatial partitions or areal units (such as regions or districts) as in, for example, population density or illness rates.



Source: https://en.wikipedia.org/wiki/Modifiable_areal_unit_problem

Lab: MAUP

- Can you imagine how does MAUP affect policy making?
- List three examples:

1) ...
2) ...
3) ...

Spatial Interpolation

- The assumption that makes interpolation a viable option is that spatially distributed objects are spatially correlated; in other words, things that are close together tend to have similar characteristics.
- For instance, if it is raining on one side of the street, you can predict with a high level of confidence that it is raining on the other side of the street. You would be less certain if it was raining across town and less confident still about the state of the weather in the next county.

Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/understanding-interpolation-analysis.htm

Spatial Interpolation

- **Interpolating a rainfall surface**
- The input here is a point dataset of known rainfall-level values, shown by the illustration on the left.

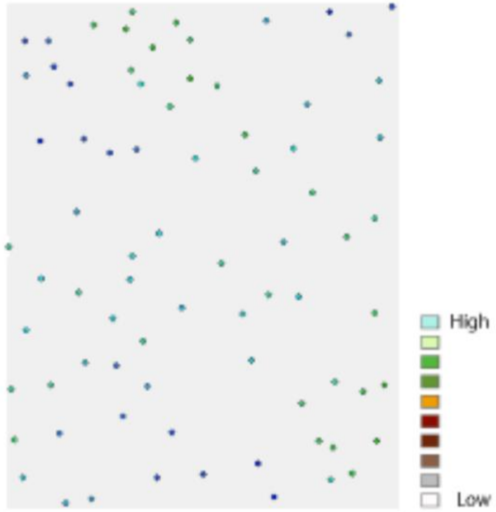
Input rainfall point data

Interpolated rainfall surface

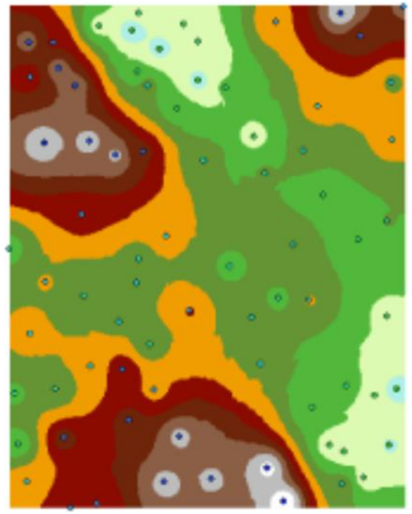
Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/understanding-interpolation-analysis.htm

Spatial Interpolation

- **Interpolating an elevation surface**
- A typical use for point interpolation is to create an elevation surface from a set of sample measurements.



Input elevation point data



Interpolated elevation surface

Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/understanding-interpolation-analysis.htm

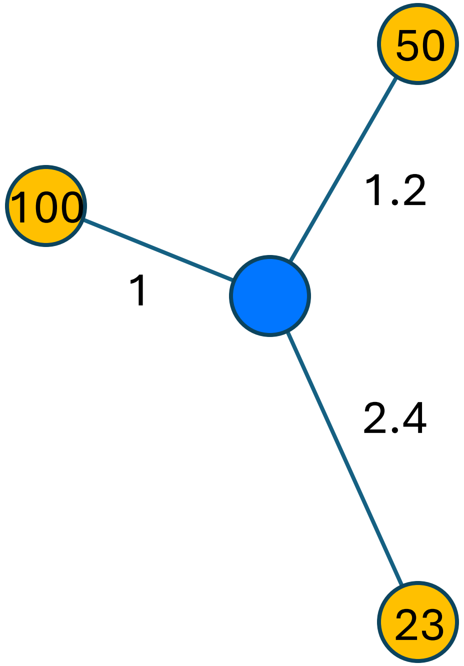
Inverse Distance Weighting

- The IDW (Inverse Distance Weighted) tool uses a method of interpolation that estimates cell values by averaging the values of sample data points in the neighborhood of each processing cell. The closer a point is to the center of the cell being estimated, the more influence, or weight, it has in the averaging process.

Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/comparing-interpolation-methods.htm#ESRI_SECTION1_6A0EECC499AA4BB191961A99AFA9352F

Inverse Distance Weighting

- Mathematical Version:



$$\widehat{v_1} = \frac{\sum_{i=1}^{n} \frac{1}{d_i^p} v_i}{\sum_{i=1}^{n} \frac{1}{d_i^p}}$$

$$z(x) = \frac{\sum w_i z_i}{\sum w_i} = \frac{\frac{100}{1^2} + \frac{50}{1.2^2} + \frac{23}{2.4^2}}{\frac{1}{1^2} + \frac{1}{1.2^2} + \frac{1}{2.4^2}}$$

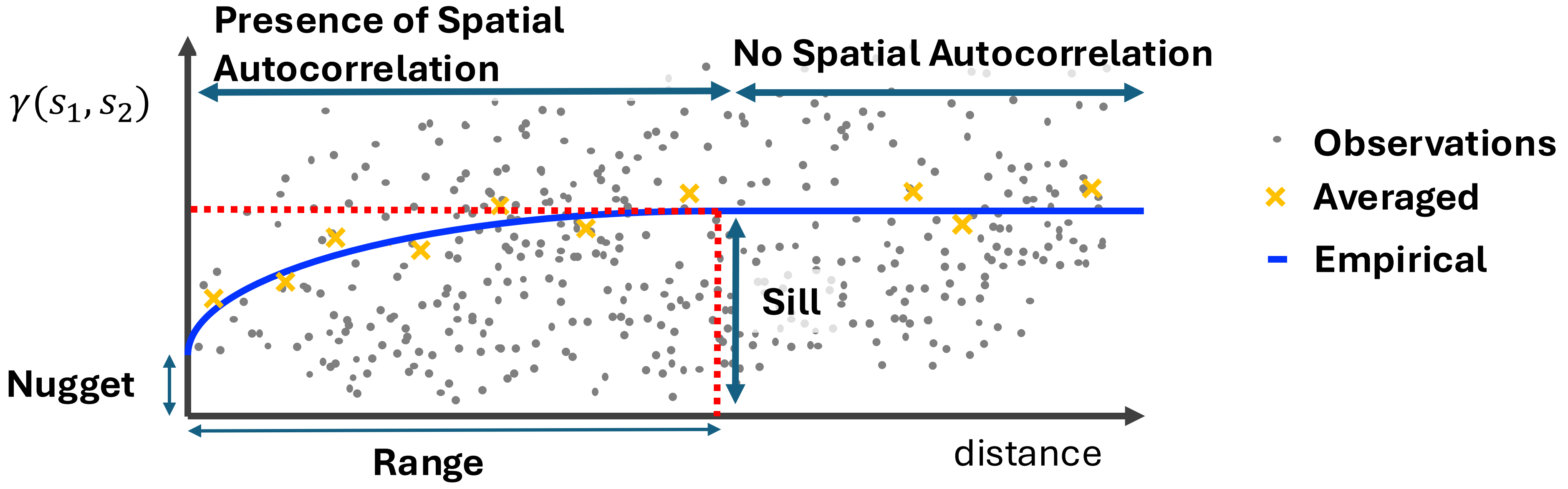
Kriging

- Kriging is an advanced geostatistical procedure that generates an estimated surface from a scattered set of points with z-values. More so than other interpolation methods, a thorough investigation of the spatial behavior of the phenomenon represented by the z-values should be done before you select the best estimation method for generating the output surface.

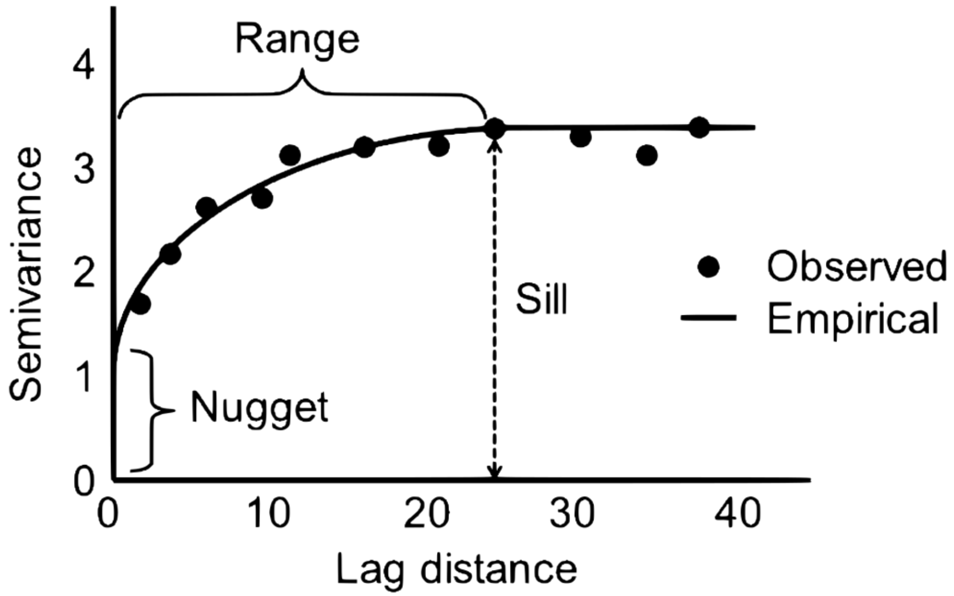
Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/comparing-interpolation-methods.htm#ESRI_SECTION1_6A0EECC499AA4BB191961A99AFA9352F

Kriging – Variogram

- In spatial statistics, the theoretical variogram, denoted $2\gamma(s_1, s_2)$, is a function describing the degree of spatial dependence of a spatial random field or stochastic process $Z(s)$.
- The semivariogram $\gamma(s_1, s_2)$ is half the variogram.



Kriging – Concept



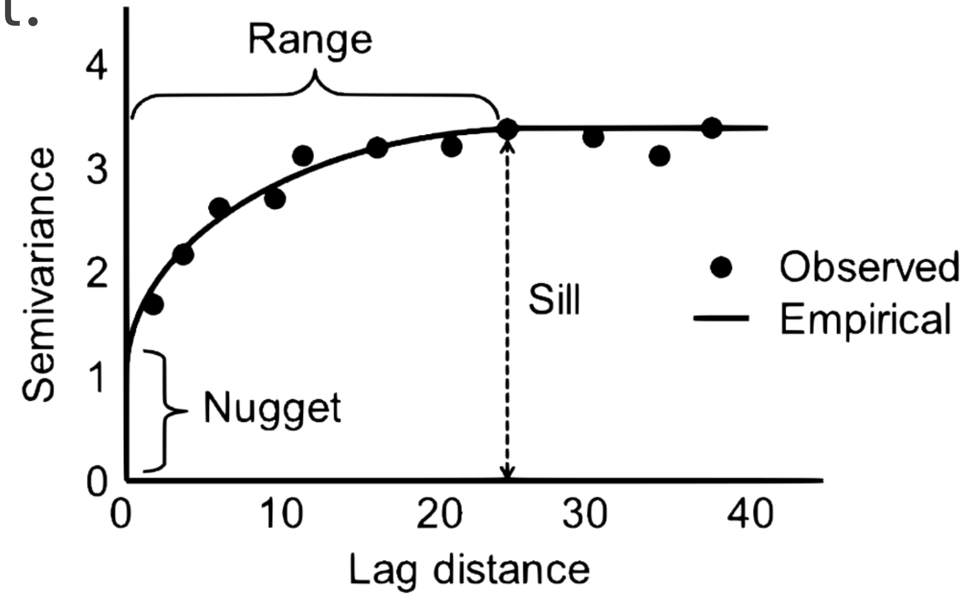
- In summary, the following parameters are often used to describe variograms:
- **Nugget** (n): The height of the jump of the semivariogram at the discontinuity at the origin.
- **Sill** (s): Limit of the variogram tending to infinity lag distances.
- **Range** (r): The distance in which the difference of the variogram from the sill becomes negligible. In models with a fixed sill, it is the distance at which this is first reached; for models with an asymptotic sill, it is conventionally taken to be the distance when the semivariance first reaches 95% of the sill.

Source: https://en.wikipedia.org/wiki/Variogram

El-Sinawi, A., Awadallah, M., & Janajreh, I. (2018). Damage Identification of HAWT Blade using Ordinary Linear Kriging Method and Variation of Blade's Modal Parameters. *Int. J. of Thermal & Environmental Engineering*, *17*(1), 59-66.

Kriging – Variogram

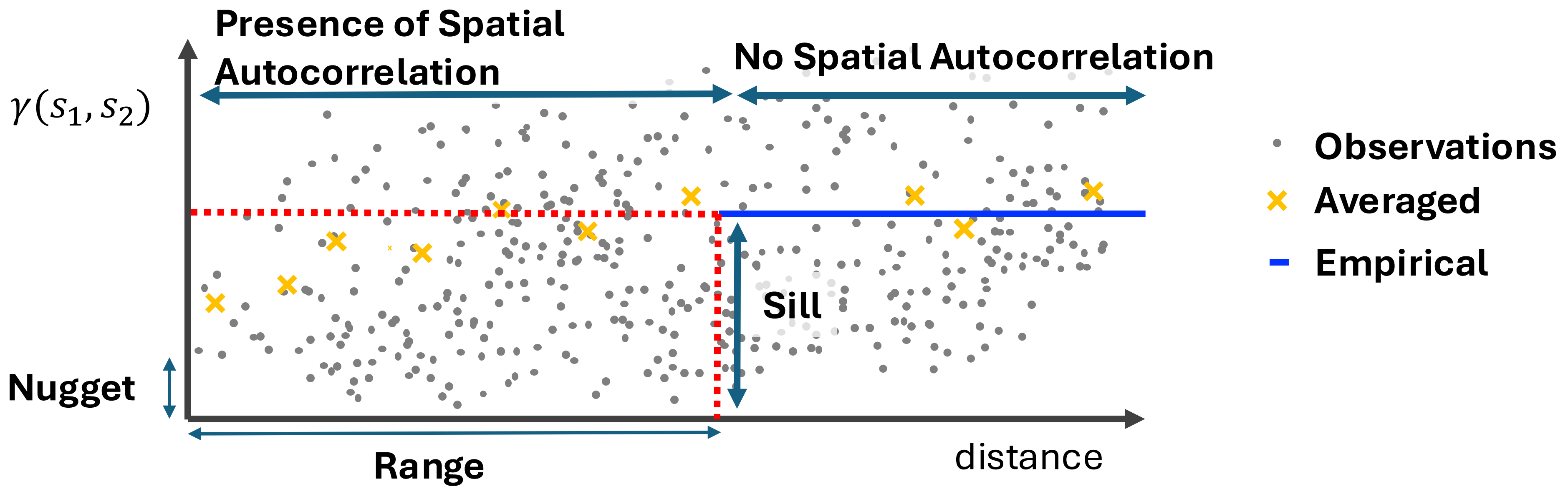
- A **variogram** might be thought of as "dissimilarity between point values as a function of distance", such that the dissimilarity is greater for points that are farther apart.



- $2\gamma(h) = E\{[Z(x+h) - Z(x)]^2\}$
- $2\gamma(h) = average\left[\left(Z(i) - Z(j)\right)^2\right]$
- $2\hat{\gamma}(h) = \frac{1}{N(h)} \sum_{N(h)} \left(Z(s_i) - Z(s_j)\right)^2$
- $N(h)$: the number of paired comparisons at lag distance h.

Kriging – Semivariogram

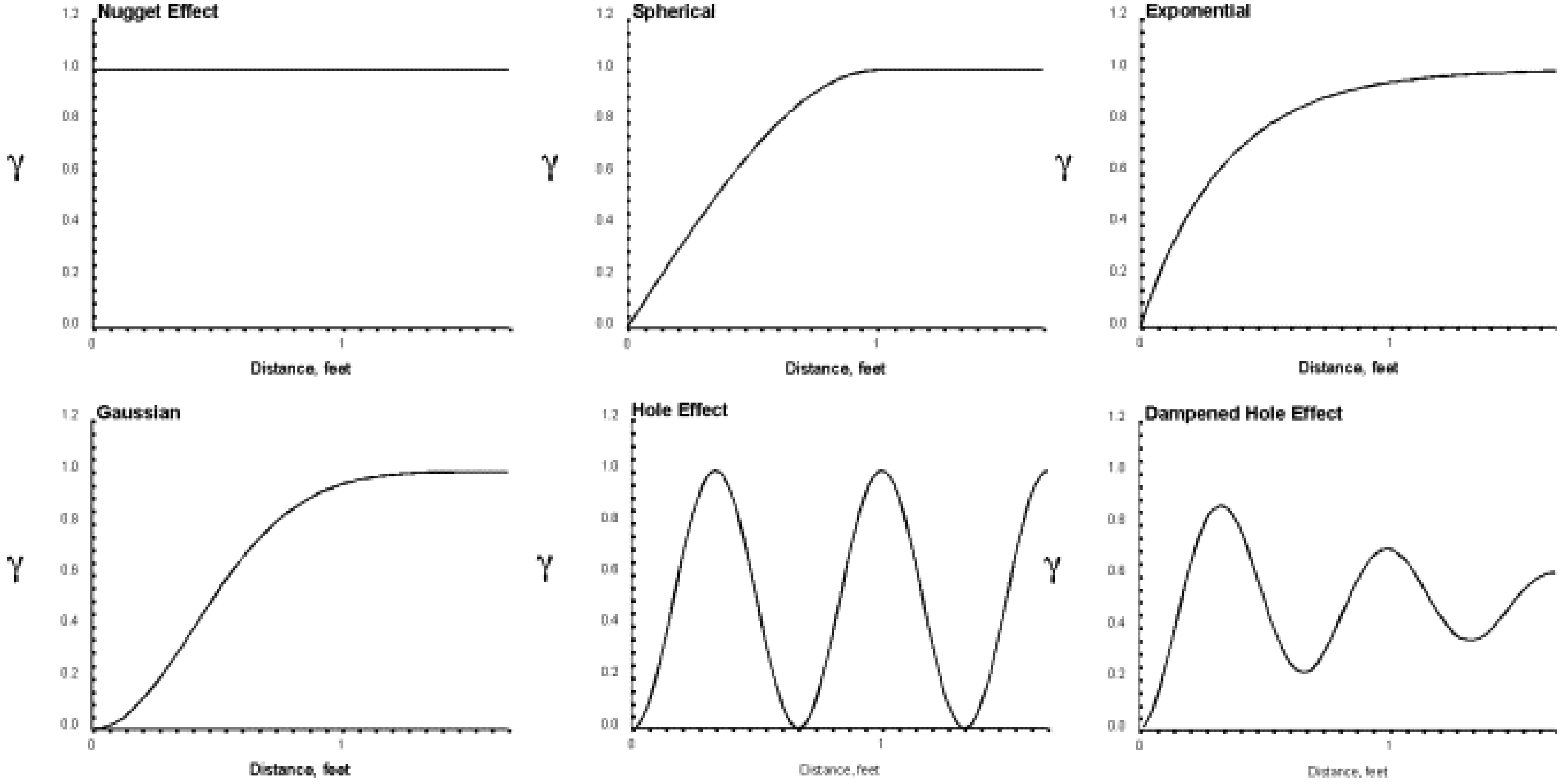
- **Semivariogram**
- $\hat{\gamma}(h) = \frac{1}{2N(h)} \sum_{i=1}^{N(h)} [Z(x_i + h) - Z(x_i)^2]$



Kriging – Semivariogram Properties

- According to (Cressie 1993, Chiles and Delfiner 1999, Wackernagel 2003) the theoretical variogram has the following properties:
    1) The semivariogram is nonnegative $\gamma(s_1, s_2) \geq 0$, since it is the expectation of a square.
    2) The semivariogram $\gamma(s_1, s_2) = \gamma_i(0) = E\left((Z(s_1) - Z(s_1))^2\right) = 0$ at distance 0 is always 0, since $Z(s_1) - Z(s_1) = 0$.
    3) If the covariance function of a stationary process exists, it is related to variogram by $2\gamma(s_1, s_2) = C(s_1, s_1) + C(s_2, s_2) + 2C(s_1, s_2)$
    4) If a stationary random field has no spatial dependence (i.e. $C(h) = 0$ if $h \neq 0$), the semivariogram is the constant $var\left(Z(s)\right)$ everywhere except at the origin, where it is zero.

Kriging – Variogram Fitting



Kriging – Stationary Models

Stationary Models	Mathematical Definition	Note
Exponential models	$$\gamma(d) = \sigma^2\left[1 - \exp\left(-\frac{d}{L}\right)\right]$$	Integral length: $L\ (> 0)$ $\alpha \approx 3L$
Spherical model	$$\gamma(d) = \begin{cases} \sigma^2\left[\frac{3}{2}\frac{d}{\alpha} - \frac{1}{2}\left(\frac{d}{\alpha}\right)^3\right], if\ d \leq \alpha \\ \sigma^2, if\ d > \alpha \end{cases}$$	α: affected region
Gaussian model	$$\gamma(d) = \sigma^2\left[1 - \exp\left(-\frac{d^2}{L^2}\right)\right]$$	$\alpha \approx 1.73L$
Inverse-distance model	$$\gamma(d) = \sigma^2\left[1 - \frac{L}{\sqrt{d^2 + L^2}}\right]$$	$\alpha \approx 20L$
Hole-effect model	$$\gamma(d) = \sigma^2\left[1 - \left(1 - \frac{d}{L}\right)\exp\left(-\frac{d}{L}\right)\right]$$	$d = 0.88L$ $\rho = 0.05$

Kriging – Non-Stationary Models

Stationary Models	Mathematical Definition	Note
Power models	$\gamma(d) = \theta \cdot d^s$	Integral length: L (> 0) $\alpha \approx 3L$
Linear model	$\gamma(d) = \theta \cdot d$	α: affected region
Logarithmic model	$\gamma(d) = \theta \cdot \log\left(1 + \frac{d}{L}\right)$	$\alpha \approx 1.73L$

Kriging – Non-Stationary Models

- If nuggest exists, …
- $\gamma(d) = C_0\delta(d) = \begin{cases} C_0 & if > 0 \\ 0 & if\ d = 0 \end{cases}$

Stationary Models	Mathematical Definition	Note
Power models	$\gamma(d) = C_0 + (\sigma^2 - C_0)\left[1 - \exp\left(-\frac{d}{L}\right)\right]$	
Spherical model	$\gamma(d) = C_0 + (\sigma^2 - C_0)\left[\frac{3}{2}\frac{d}{\alpha} - \frac{1}{2}\left(-\frac{d}{\alpha}\right)^3\right]$	for $0 < d <= \alpha$
Gaussian model	$\gamma(d) = C_0 + (\sigma^2 - C_0)\left[1 - \exp\left(-\frac{d}{L}\right)^2\right]$	

Kriging – Non-Stationary Models

Stationary Models	Mathematical Definition	Note
Inverse weight models	$\gamma(d) = C_0 + (\sigma^2 - C_0)\left[1 - \frac{L}{\sqrt{d^2 + L^2}}\right]$	
Nugget effect model	$\gamma(d) = C_0 + (\sigma^2 - C_0)\left[\frac{3}{2}\frac{d}{\alpha} - \frac{1}{2}\left(-\frac{d}{\alpha}\right)^3\right]$	for $0 < d <= \alpha$
Exponential model	$\gamma(d) = C_0 + \theta\left(\frac{d}{L}\right)$	
Linear model	$\gamma(d) = C_0 + \theta \cdot d$	
Logarithmic model	$\gamma(d) = C_0 + \theta \cdot \log\left(1 + \frac{d}{L}\right)$	

Natural Neighbor

- Natural Neighbor interpolation finds the closest subset of input samples to a query point and applies weights to them based on proportionate areas to interpolate a value (Sibson, 1981). It is also known as Sibson or "area-stealing" interpolation.

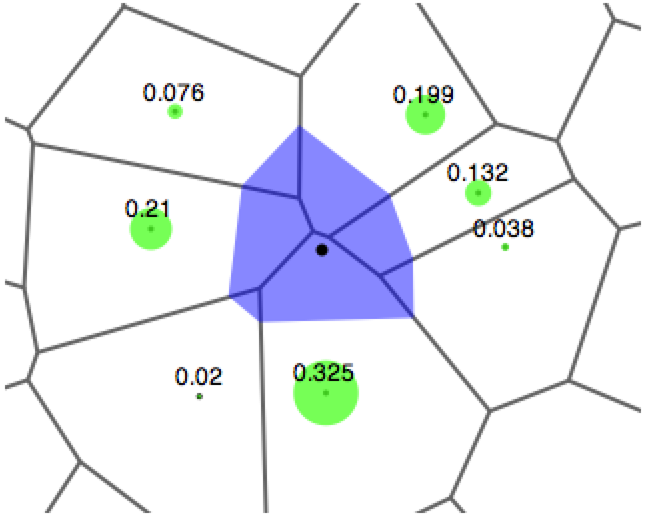
$$G(x) = \sum_{i=1}^{n} w_i(x) f(x_i)$$

Sibson weights

$$w_i(x) = \frac{A(x_i)}{A(x)}$$

Laplace weights

$$w_i(x) = \frac{\frac{l(x_i)}{d(x_i)}}{\sum_{k=1}^{n} \frac{l(x_k)}{d(x_k)}}$$



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Spline

- The Spline tool uses an interpolation method that estimates values using a mathematical function that minimizes overall surface curvature, resulting in a smooth surface that passes exactly through the input points.

$$f(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x_1 - x_0), x_0 \leq x \leq x_1$$

$$= f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x_2 - x_1), x_1 \leq x \leq x_x$$

$$\dots$$

$$= f(x_{n-1}) + \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}(x_n - x_{n-1}), x_{n-1} \leq x \leq x_n$$

$$\Rightarrow \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}$$

Source: https://pro.arcgis.com/en/pro-app/latest/tool-reference/spatial-analyst/comparing-interpolation-methods.htm#ESRI_SECTION1_6A0EECC499AA4BB191961A99AFA9352F

The End

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